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Note

On the chromatic number of $\mathbb{R}^n$ with an arbitrary norm

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ABSTRACT

In this work we study the Nelson–Erdös–Hadwiger problem on coloring metric spaces. Let $\chi(\mathbb{R}^n_K)$ be the chromatic number of the space $\mathbb{R}^n$ with an arbitrary norm determined by a centrally symmetric convex body K. Füredi and Kang (2008) [5] proved that $\chi(\mathbb{R}^n_K) \leq (5+o(1))^n$. We improve this bound to $\chi(\mathbb{R}^n_K) \leq (4+o(1))^n$ in the general case, and further improvements are obtained in the case of l_p-norms.

1. Introduction

The *chromatic number of a plane* is the smallest number of colors needed to color the points of the plane such that any two points at unit distance apart are colored by different colors. The problem of finding the chromatic number of a plane was posed by Nelson in 1950 (see the history of the problem in [11]). This problem can be generalized as follows.

Let (Γ, ρ) be a metric space. The *chromatic number* of (Γ, ρ) is the minimal number m such that there is a partition $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_m$ satisfying the following condition: for each i, $i = 1, \ldots, m$, and for any pair of distinct points x and y from the part Γ_i, the norm $\rho(x, y) \neq 1$.

We denote the chromatic number of the space $\mathbb{R}^n$ with Euclidean norm by $\chi(\mathbb{R}^n)$, the chromatic number of the space $\mathbb{R}^n$ with l_p-norm by $\chi(\mathbb{R}^n_p)$, and the chromatic number of the space $\mathbb{R}^n$ with the norm determined by a centrally symmetric convex body K, by $\chi(\mathbb{R}^n_K)$. In particular, if K is the unit sphere S^{n-1}, then $\chi(\mathbb{R}^n_K) = \chi(\mathbb{R}^n_2) = \chi(\mathbb{R}^n)$.

The exact values of chromatic numbers in most cases are unknown. Even in the classical case of the plane $\mathbb{R}^2$, there is only a simple estimation $4 \leq \chi(\mathbb{R}^2) \leq 7$.

On the other hand, during the last sixty years since Nelson stated the problem, many substantial results have been obtained (see [9,2,12]). Chromatic numbers were studied for colorings of special types. For instance, Székely [12] considered the colorings where every color is a measurable set or every color is a union of disjoined polyhedra. Also, the colorings with multiple "forbidden" distances were considered in [12,6]. The case of arbitrary metric spaces was introduced by Benda and Perles in [1].

For the Euclidean norm, the best asymptotic upper and lower bounds were found by Larman and Rogers [8] and by Raigorodskii [9], respectively:

$$\big(1.239 + o(1)\big)^n \leq \chi(\mathbb{R}^n) \leq \big(3 + o(1)\big)^n.$$

In other words, the function $\chi(\mathbb{R}^n)$ grows exponentially with n.

Also, an exponential growth was shown for the chromatic number of the l_p-space $\mathbb{R}^n_p$ in [6]. However, there is no exponential lower bound on $\chi(\mathbb{R}^n_K)$ for the space $\mathbb{R}^n_K$ with an arbitrary norm determined by a body K.

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Recently, Füredi and Kang [5] proved the following inequality:

$$\chi(\mathbb{R}^n_K) \le cn(\ln n)5^n,$$

where the constant c does not depend on K. In the following theorem we improve this bound.

Theorem 1. *Let K be an arbitrary centrally symmetric convex body. Then*

$$\chi(\mathbb{R}^n_K) \le \frac{\big(\ln n + \ln\ln n + \ln 4 + 1 + o(1)\big)}{\ln\sqrt{2}} \cdot 4^n.$$

A further improvement of this theorem can be obtained in the case of the space $\mathbb{R}^n_p$.

Theorem 2. *For all $p > 2$ there exist c_p, $c_p < 1$, and δ_n, $\delta_n \to 0$ as $n \to \infty$, such that*

$$\chi(\mathbb{R}^n_p) \le 2^{(1+c_p+\delta_n)n}.$$

The same inequality holds also under the additional assumption that $c_p \to 0$ as $p \to \infty$. Moreover, if $\omega(n)$ is a function such that $\omega(n) \to \infty$, as $n \to \infty$, and $p(n) > \omega(n) n \ln\ln n$, then

$$\chi(\mathbb{R}^n_{p(n)}) \le \big(\ln n + \ln\ln n + \ln 2 + 1 + o(1)\big) \cdot n \cdot 2^n.$$

Theorem 2 implies that $\chi(\mathbb{R}^n_p) \le (2+\gamma_{n,p})^n$, where $\gamma_{n,p} \to 0$ as $n, p \to \infty$. On the other hand, it is easy to see that $\chi(\mathbb{R}^n_\infty) = 2^n$. Hence, at least the exponent base in the upper bound for $\chi(\mathbb{R}^n_p)$ becomes closer to 2, the exponent base in the exact value of the chromatic number of $\mathbb{R}^n_\infty$.

2. Proofs of the results

In 2.1 we formulate Theorem 3, which follows directly from [4]. An independent and shorter proof of the theorem is given in 2.3. In 2.2 we prove Theorems 1 and 2 using Theorem 3.

2.1. Lattice coverings of spaces

Let $\|x\|_K$ be the norm in $\mathbb{R}^n$ induced by a centrally symmetric convex body K, and let Λ be a lattice. We denote the volume K by V, and the determinant of the lattice Λ by $\det \Lambda$.

Let X denote the disjoint union $\bigsqcup_{q\in\Lambda}(K+q)$. We assume that the copies of K corresponding to different points of the lattice do not intersect each other, i.e. they form a packing. Let $\delta(X) = V/\det\Lambda$ be the density of the set X.

Theorem 3. *Suppose that $\delta(X) = \big(c+\varphi(n)\big)^{-n}$, where c is a constant, $c > 1$, and $\varphi(n) \to 0$ as $n \to \infty$. Then there exists a covering of the space by $z = z_\psi$ copies of the set X such that $z = n\cdot\delta^{-1}(X)\cdot\big(\ln n + \ln\ln n + \ln c + 1 + \psi(n)\big)$ for some function ψ satisfying $\psi(n) \to 0$ as $n \to \infty$.*

2.2. Proofs of Theorems 1 and 2

For the proofs, we need the following theorem from [10]. Let B be a bounded Borel measurable set in $\mathbb{R}^n$ containing the origin, and let $V(B)$ be the volume of B. For the set $\Upsilon(B)$ of all lattices intersecting B only at the origin, we denote the infimum of $\det\Lambda$, $\Lambda \in \Upsilon(B)$, by $\Delta(B)$.

Theorem 4 (Schmidt)**.** *For sufficiently large n and some constant c the following inequality holds: $V(B)/\Delta(B) > n\ln\sqrt{2} - c$.*

Proof of Theorem 1. We apply Theorem 4 to the body K that determines the norm in $\mathbb{R}^n_K$. Since K is convex, centrally symmetric, and compact, there exists a lattice Λ such that $\det\Lambda = \Delta(B)$ (see [7]). The sets $\frac{1}{2}K + g$, $g \in \Lambda$, form a packing X with the density $\delta(X) \ge (n\ln\sqrt{2} - c)2^{-n}$ (for more details, see Sections 19 and 20 of [7]).

Let $f(n)$ be a function such that $(a + f(n))^n = a^n + o\left(\frac{1}{n^n}\right)$ for any $a > 0$. Consider the set

$$\Theta = \bigcup_{g\in\Lambda}\left(\frac{1-f(n)}{4}K + g\right).$$

The diameter of the set $\frac{1-f(n)}{4}K$ is less than $1/2$. On the other hand, the distance between the points $x_1 \in \frac{1-f(n)}{4}K + g_1$ and $x_2 \in \frac{1-f(n)}{4}K + g_2$ is greater than $1/2$ if $g_1 \ne g_2$. Therefore, the distance between two points of Θ cannot be $1/2$, and we can paint the set Θ with one color. In addition,

$$\delta(\Theta) = \left(\frac{1-f(n)}{4}\right)^n (n\ln\sqrt{2}-c) = 4^{-n}(n\ln\sqrt{2}-c) + o\left(\frac{1}{n^n}\right) = \big(4+\varphi(n)\big)^{-n}$$

for some $\varphi(n) = o(1)$.

Applying Theorem 3, we obtain a covering of the space by not more than

$$\frac{n\big(\ln n + \ln\ln n + \ln 4 + 1 + \psi(n)\big)}{4^{-n}(n\ln\sqrt{2}-c) + o\left(\frac{1}{n^n}\right)} = \frac{\big(\ln n + \ln\ln n + \ln 4 + 1 + o(1)\big)}{\ln\sqrt{2}} \cdot 4^n$$

copies of Θ, and each copy can be painted with one color. $\square$

Proof of Theorem 2. The proof is similar to the proof of Theorem 1, but initially we take the packing constructed by Elkies et al. in [3]. It is a set Θ' of balls in $\mathbb{R}^n_p$ with the density $\delta(\Theta') = 2^{-c_p n + o(1)}$, where $c_p < 1$ when $p > 2$, and $c_p \sim \frac{\ln\ln p}{p\ln 2}$ as $p \to \infty$. In this case, the function δ_n, which appears in the statement of Theorem 2, has the following form:

$$\delta_n = \frac{\ln\big(n\ln n + n\ln\ln n + (\ln c + 1)n\big) + o(1)}{n}.$$

In particular, if $\omega(n)$ is a function that tends to infinity arbitrarily slowly, and $p > \omega(n) n \ln\ln n$, then $c_p n = o(1)$ and $\delta(\Theta') = 1 - o(1)$ as $n \to \infty$. Proceeding as in the proof of Theorem 1, we obtain the result. $\square$

2.3. Proof of Theorem 3

Consider z_ψ randomly chosen points x_i, which are uniformly distributed in the fundamental parallelepiped of the lattice Λ. Let $\eta = \frac{1}{n\ln n}$, and let S be a set, which is not covered by the set

$$\Psi_\eta = \bigcup_{i=1}^{z_\psi} \bigsqcup_{q\in\Lambda} \big((1-\eta)K + x_i + q\big).$$

The expectation of the density of S is $E(\delta(S)) = \big(1-(1-\eta)^n\delta(X)\big)^{z_\psi}$. Then

$$\begin{aligned}
\ln E(\delta(S)) &= z_\psi \ln\left(1-\delta(X)\left(1-\frac{1}{n\ln n}\right)^n\right) \le -z_\psi\delta(X)\left(1-\frac{1}{n\ln n}\right)^n \\
&= -n(\ln n + \ln\ln n + \ln c + 1 + \psi(n))\left(1-\frac{1}{\ln n} + O\left(\left(\frac{1}{\ln n}\right)^2\right)\right) \\
&= -n\left(\ln n + \ln\ln n + \ln c + \psi(n) - O\left(\frac{\ln\ln n}{\ln n}\right)\right).
\end{aligned}$$

It is easy to see that for any function $\varphi(n)$, such that $\varphi(n) \to 0$ as $n \to \infty$, there exists a function $\psi(n)$, such that $\psi(n) \to 0$ as $n \to \infty$, and for each n

$$\ln n + \ln\ln n + \ln c + \psi(n) - O\left(\frac{\ln\ln n}{\ln n}\right) > \ln n + \ln\ln n + \ln\big(c+\varphi(n)\big).$$

Let $\psi(n)$ be such a function, and let $z = z_\psi$. Then we have the following inequality:

$$E(\delta(S)) = \left(\frac{1}{e^{\ln c}\cdot e^{\psi(n)}\cdot e^{-O\left(\frac{\ln\ln n}{\ln n}\right)}\cdot n\ln n}\right)^n < \left(\frac{1}{(c+\varphi(n))n\ln n}\right)^n = \eta^n\delta(X).$$

Hence, there is a set of points $x_1, \dots, x_z$ such that $\delta(S) < \eta^n\delta(X)$. We will show that the set $\Psi = \bigcup_{i=1}^{z}(X + x_i)$ satisfies the conclusion of the theorem. Let us assume the contrary. Then there exists a point $x \notin \Psi$. This implies that for all points of the form $y = x_i + q$, $q \in \Lambda$, we have $\|x-y\|_K > 1$. Then, by the triangle inequality, $\|x'-y\|_K > 1-\eta$ for all $x' \in \eta K + x + q'$, $q' \in \Lambda$, and for all $y = x_i + q$, $q \in \Lambda$. Hence, $(\eta K + x + q) \cap \Psi_\eta = \varnothing$ for every $q \in \Lambda$. This implies that S contains the set $\bigsqcup_{q\in\Lambda}(\eta K + x + q)$, and $\delta(S) \ge \eta^n\delta(X)$, a contradiction. $\square$

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